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GENERAL INSTABILITY OF STIFFENED CYLINDERS

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SUMMARY

Theoretical buckling stresses are determined in explicit form for circular cylinders with circumferential and axial stiffening. The loadings are axial compression, radial pressure, hydrostatic pressure, and torsion. Analyses were confined to moderate-length and long cylinders. The investigation was based upon the use of a form of Donnell's equation derived by Taylor which is applicable to orthotropic cylinders. The derivation of this equation is presented in this report.

INTRODUCTION

General instability in stiffened circular cylinders has been investigated for several cases of loading and types of stiffening. Analyses were performed utilizing energy methods, the three differential equations for orthotropic cylinders of Flügge (ref. 1), and various forms of Donnell's equation (ref. 2) applicable to orthotropic cylinders.

Except for the analyses by Dschou (ref. 3) and Taylor (ref. 4), these investigations have not yielded explicit solutions for buckling stress. Furthermore, the analyses for external-pressure loadings have been confined to ring-stiffened cylinders. In the present report, explicit expressions are presented for axial-compression, external-pressure, and torsional loadings on moderate-length and long cylinders with both axial and circumferential stiffening.

The axial-load solution is that developed by Taylor, who utilized the simplifications employed by Donnell, together with the effects of orthotropicity, and derived an eighth-order partial differential equation for axial-compression buckling. In this report, the equation of Taylor has been extended to include torsional and pressure loadings. In addition, Taylor's assumption of a zero value for Poisson's ratio has been retained.

The analyses are confined to moderate-length cylinders, for which the boundary conditions influence the buckling stress, and to long cylinders, for which there is no boundary influence. Details of boundary conditions for the different loadings are discussed in the pertinent sections below.

The investigation is restricted to elastic buckling. Linear theory is used in all cases. Section properties of the frames, stiffeners, and sheet are nominal, as depicted in figure 1. The actual properties to be used in design are discussed in part VI of the Handbook of Structural Stability (ref. 5). This handbook contains a critical review of the field of general instability. It presents comparisons of theory and test data which delineate the utility both of the theoretical results presented herein and of other theories which do not employ the orthotropic-shell theory.

A summary of the results of the investigations appears in table 1. Short discussions of the analyses of each case investigated are also included. The derivations of the general instability stresses are presented in appendixes A to D.

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SYMBOLS

A_f	distributed area of frame including effective sheet, in., $\bar{A}_f/d$
$\bar{A}_f$	area of frame including effective sheet, sq in.
A_s	distributed area of stiffener including effective sheet, in., $\bar{A}_s/b$
$\bar{A}_s$	area of stiffener including effective sheet, sq in.
a	constant
b	stiffener spacing, in.
d	frame spacing, in.
E	Young's modulus, psi
F	Airy's stress function
G	shear modulus, psi
I_f	distributed bending moment of inertia of frame, $\bar{I}_f/d$, cu in.
I_s	distributed bending moment of inertia of stiffener, $\bar{I}_s/b$, cu in.

$\bar{I}_f$ bending moment of inertia of frame including effective sheet, in.^{4}

$\bar{I}_s$ bending moment of inertia of stiffener including effective sheet, in.^{4}

$J = J_f + J_s$

J_f distributed torsional moment of inertia of frame, $\bar{J}_f/d$, cu in.

J_s distributed torsional moment of inertia of stiffener, $\bar{J}_s/b$, cu in.

$\bar{J}_f$ torsional moment of inertia of stiffener, in.^{4}

$\bar{J}_s$ torsional moment of inertia of frame, in.^{4}

k_y buckling coefficient for unstiffened circular cylinder under lateral pressure

k_p buckling coefficient for unstiffened circular cylinder under hydrostatic pressure

L length of cylinder, in.

$$M_x = EI_s \frac{\partial^2 w}{\partial x^2}$$

$$M_{xy} = -\frac{1}{2} GJ \frac{\partial^2 w}{\partial x\, \partial y}$$

$$M_y = EI_f \frac{\partial^2 w}{\partial y^2}$$

m,n parameters of axial and circumferential buckle wave lengths

$\bar{m} = mL/\pi$

N_x axial normal load on cylinder, $\sigma_x t_s$, lb/in.

N_{xy} shear load on cylinder, τt, lb/in.

N_y circumferential normal load on cylinder, $\sigma_y t_f$, lb/in.

p pressure loading on cylinder, psi

Q_x, Q_y transverse shears on shell element, lb

R radius of cylinder, in.

T torque loading on cylinder wall, in.

t thickness of cylinder wall, in.

$\bar{t}$ effective thickness of cylinder wall in shear-buckled state

t_f distributed area of frame, A_f/d, in.

t_s distributed area of stiffener, A_s/b, in.

u,v,w displacements in x-, y-, and z-directions, in.

w_{mn} deflection constant

x axial coordinate, in.

y circumferential coordinate, in.

z radial coordinate, in.

Z_L parameter for unstiffened cylinder, L^2/Rt when $\nu = 0$

$\beta = n/m$

γ shear strain

ϵ_x axial normal strain

ϵ_y tangential normal strain

λ half-wave length of buckle in circumferential direction, in.

ν Poisson's ratio

ρ_f radius of gyration of frame section, $\rho_f^2 = \bar{I}_f/A_f = I_f/t_f$, in.

σ_c axial-compressive-buckling stress, psi

σ_x axial normal stress, psi

σ_y circumferential-compressive-buckling stress; also, general circumferential normal stress, psi

τ shear-buckling stress; also, general shear stress, psi

HISTORICAL BACKGROUND

Early theoretical investigations into the buckling of stiffened circular shells were performed by Flügge (ref. 1), Dschou (ref. 3), and Taylor (ref. 4). Flügge derived the trio of linear equilibrium equations analogous to those for isotropic cylinders which have been utilized by many investigators. Dschou solved these equations for stiffened circular cylinders under axial load.

Taylor derived a differential equation for axially loaded orthotropic circular cylinders utilizing the same approach as did Donnell in obtaining his well-known eighth-order partial differential for isotropic circular cylinders (ref. 2). The result is a relatively simple explicit solution for the general instability stress which reduces to the classical result for an isotropic cylinder. There is a slight discrepancy due to Taylor's assumption of a zero value for Poisson's ratio.

Buckling of ring-stiffened circular cylinders under external pressure was investigated by Salerno and Levine (ref. 6) and by Kendrick (ref. 7) by using the energy approach. Kendrick's calculations led to higher instability stresses than those obtained by Salerno and Levine, who omitted certain terms in the energy equations.

Bodner investigated this case (ref. 8) using an equation of the Donnell type similar to that presented in this report in which Poisson's ratio is included. The complete length range was analyzed.

Stein, Sanders, and Crate determined the section properties of ring stiffeners required to avoid general instability in cylinders loaded in torsion (ref. 9). Hayashi (ref. 10) analyzed stiffened cylinders in torsion employing the identical approach used by Donnell for isotropic cylinders. Hayashi showed that Donnell's results are a limiting case of the buckling of orthotropic cylinders in torsion. The complete length range is included in this treatment, and the results are obtained in the same form as found by Donnell, except for the more general form of the parameters.

THEORETICAL RESULTS

The theoretical buckling stresses for orthotropic cylinders under various loadings are shown in table 1. Derivations of these expressions are presented in the appendices. Buckling stresses were determined on the assumption that the spacings of the longitudinal stiffeners and the circumferential frames were small enough to consider the cylinder to act as a uniform orthotropic shell. Effects of boundary conditions on the general instability stresses for the various loadings are discussed in the following paragraphs.

Axial Load

The solution chosen by Taylor for the case of axial loading represents the waveform assumed in the classical solution of simply supported isotropic cylinders, in which linear theory is used. It is applicable to both moderate-length and long cylinders.

External Pressure

The solution for external pressure is identical to that for axial load and was applied by Batdorf to external radial and hydrostatic pressures on simply supported cylinders (ref. 11). For long isotropic cylinders Donnell's equation leads to a buckling stress that is too high, compared with the results of an exact analysis, by a factor of 4/3. Although the corresponding result is obtainable for orthotropic cylinders by the method presented in this report, Levy's result for radially loaded rings (ref. 12) has been presented instead since it corresponds to the exact solution.

The analysis is simplified for external-pressure loadings by assuming that the ratio of axial to circumferential wave length of the buckle is negligibly small compared with unity. This simplification is justifiable for moderate-length cylinders. Furthermore, the explicit result agrees with Bodner's data for ring-stiffened cylinders in this length range.

Torsion

The solutions and simplifications applied to stiffened cylinders loaded in torsion were employed by Gerard and Becker for moderate-length isotropic cylinders (ref. 13). The long-cylinder solution does not satisfy boundary conditions, although it apparently is a satisfactory representation of the buckle waveform for isotropic cylinders. However, in this length range a satisfactory solution to the problem may be obtained although boundary conditions are not satisfied for isotropic cylinders. The same situation was assumed to apply to stiffened cylinders.

The moderate-length solution satisfies the requirement that $w = 0$ on the boundaries but does not correspond to vanishing moments at these locations. However, the buckling stress for an isotropic moderate-length cylinder in torsion is relatively insensitive to specifications on these latter quantities, since there is less than 10-percent difference between the theoretical buckling stresses for simply supported and clamped edges.

The general instability behavior of orthotropic cylinders is considered to parallel the buckling behavior of moderate-length and long isotropic circular shells. Consequently, the same solution to the buckling equation and the same simplifications in the mathematics are assumed to be applicable. The mathematics was simplified by assuming that the ratio of circumferential to axial wave length was negligible compared with unity.

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APPENDIX A

DERIVATION OF DIFFERENTIAL EQUATION

The derivation of Taylor's differential equation in which the effects of shear and circumferential normal stress are also included is presented below. The geometric properties of a stiffened cylinder are portrayed schematically in figure 1.

Taylor derived the equilibrium equation for an orthotropic cylinder by combining the compatibility equation for forces in the plane of a plate element with that for equilibrium of forces normal to a plate element with initial curvature in one direction. These two situations are depicted in figure 2.

For equilibrium in the "plane" of the element

$$\left.\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \\ \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \end{aligned}\right\} \tag{1}$$

and for equilibrium normal to the element

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x\,\partial y} + N_y\left(\frac{1}{R} + \frac{\partial^2 w}{\partial y^2}\right) + p = 0 \tag{2}$$

in which $\frac{1}{R} + \frac{\partial^2 w}{\partial y^2}$ is the sum of the curvatures in the y-direction due to the initial shape of the shell and to the deformation under load.

It is reasonable to assume that Poisson's ratio vanishes for a stiffened cylinder fabricated in the usual manner of aircraft construction, in which case Hooke's law and the stress-strain displacement relations assume the forms

$$\frac{\sigma_x}{E} = \frac{N_x}{Et_s} = \epsilon_x = \frac{\partial u}{\partial x} \tag{3a}$$

$$\frac{\sigma_y}{E} = \frac{N_y}{Et_f} = \epsilon_y = \frac{\partial v}{\partial y} + \frac{w}{R} \tag{3b}$$

$$\frac{\tau}{G} = \frac{N_{xy}}{G\bar{t}} = \gamma = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{3c}$$

$$M_x = EI_s \frac{\partial^2 w}{\partial x^2} \tag{3d}$$

$$M_y = EI_f \frac{\partial^2 w}{\partial y^2} \tag{3e}$$

$$M_{xy} = -\frac{1}{2} GJ \frac{\partial^2 w}{\partial x\, \partial y} \tag{3f}$$

From equations (3a), (3b), and (3c), the compatibility relation

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} - \frac{\partial^2 \gamma}{\partial x\, \partial y} = \frac{1}{R} \frac{\partial^2 w}{\partial x^2} \tag{4}$$

follows, which is equivalent to

$$\frac{1}{Et_s} \frac{\partial^2 N_x}{\partial y^2} + \frac{1}{Et_f} \frac{\partial^2 N_y}{\partial x^2} - \frac{1}{G\bar{t}} \frac{\partial^2 N_{xy}}{\partial x\, \partial y} = \frac{1}{R} \frac{\partial^2 w}{\partial x^2} \tag{5}$$

If the Airy stress function F is introduced

$$\left.\begin{aligned} N_x &= \frac{\partial^2 F}{\partial y^2} \\ N_y &= \frac{\partial^2 F}{\partial x^2} \\ N_{xy} &= -\frac{\partial^2 F}{\partial x\,\partial y} \end{aligned}\right\} \tag{6}$$

then equation (5) becomes

$$\frac{1}{Et_f}\frac{\partial^4 F}{\partial x^4} + \frac{1}{G\bar{t}}\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{1}{Et_s}\frac{\partial^4 F}{\partial y^4} = \frac{1}{R}\frac{\partial^2 w}{\partial x^2} \tag{7}$$

This is the compatibility equation for forces in the plane of the element. The equilibrium equation for forces normal to the plane of the element may be obtained from equation (2) by writing the term

$$N_y \frac{1}{R} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{R}\frac{\partial^2 F}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2}$$

and utilizing equations (3d), (3e), and (3f) which yield

$$EI_s \frac{\partial^4 w}{\partial x^4} + GJ \frac{\partial^4 w}{\partial x^2 \partial y^2} + EI_f \frac{\partial^4 w}{\partial y^4} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x\,\partial y} + N_y \frac{\partial^2 w}{\partial y^2} + p = -\frac{1}{R}\frac{\partial^2 F}{\partial x^2} \tag{8}$$

The function F may easily be eliminated from equations (7) and (8), which will lead to the relation

$$\left(\frac{1}{t_f}\frac{\partial^4}{\partial x^4} + \frac{E}{G\bar{t}}\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{1}{t_s}\frac{\partial^4}{\partial y^4}\right)\left[I_s \frac{\partial^4}{\partial x^4} + \frac{GJ}{E}\frac{\partial^4}{\partial x^2 \partial y^2} + I_f \frac{\partial^4}{\partial y^4} + \right.$$

$$\left.\frac{1}{E}\left(N_x \frac{\partial^2}{\partial x^2} + 2N_{xy}\frac{\partial^2}{\partial x\, \partial y} + N_y \frac{\partial^2}{\partial y^2}\right)\right](w) + \frac{1}{R^2}\frac{\partial^4 w}{\partial x^4} = 0 \tag{9}$$

if it is assumed that p is constant. For an isotropic cylinder this will reduce to Donnell's equation for $\nu = 0$.

APPENDIX B

GENERAL INSTABILITY UNDER AXIAL COMPRESSION

Taylor utilized, as the solution to equation (9),

$$w = w_{mn} \sin mx \sin ny \tag{10}$$

for the analysis of axial compressive buckling of an orthotropic circular cylinder. For this case

$$\left.\begin{aligned} N_x &= \sigma_x t_s \\ N_y &= N_{xy} = 0 \end{aligned}\right\} \tag{11}$$

Upon solving for σ_x and minimizing with respect to m and n,

$$\sigma_c = \frac{2E}{Rt_s}\left[\frac{\beta^2 I_f + (GJ/2E)}{\left(\beta^2/t_s\right) + (E/2G\bar{t})}\right]^{1/2} \tag{12}$$

and

$$1/m = R^{1/2}\left\{\left[I_s + \beta^2(GJ/E) + \beta^4 I_f\right]\left[(1/t_f) + \left(\beta^2 E/G\bar{t}\right) + \left(\beta^4/t_s\right)\right]\right\}^{1/4} \tag{13}$$

where

$$\beta^2 = n^2/m^2 = P + \left(P^2 + Q\right)^{1/2} \tag{14}$$

where

$$P = \frac{t}{2t_f}\left[\frac{t_f I_s - t_s I_f}{t_s I_f - \left(G^2 \bar{t} J/E^2\right)}\right]$$

$$Q = \frac{t_s}{t_f}\left[\frac{t_f I_s - \left(G^2 \bar{t} J/E^2\right)}{t_s I_f - \left(G^2 \bar{t} J/E^2\right)}\right]$$

The half-wave lengths are π/m and π/n in the longitudinal and circumferential directions, respectively.

When $t_s = t_f = \bar{t} = t$ and $I_s = I_f = t^3/12$, then $\beta^2 = 1$ and equation (12) becomes

$$\sigma_c = (3)^{-1/2} Et/R \tag{15}$$

which is the classical equation for buckling of a long circular isotropic cylinder under axial load when Poisson's ratio is zero. When $\nu = 0.3$, the error is 4.6 percent.

APPENDIX C

GENERAL INSTABILITY UNDER EXTERNAL PRESSURE

For external-pressure loading, equation (9) becomes

$$\left(\frac{1}{t_f}\frac{\partial^4}{\partial x^4} + \frac{E}{G\bar{t}}\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{1}{t_s}\frac{\partial^4}{\partial y^4}\right)\left[I_s\frac{\partial^4}{\partial x^4} + \frac{GJ}{E}\frac{\partial^4}{\partial x^2 \partial y^2} + I_f\frac{\partial^4}{\partial y^4} + \right.$$

$$\left.\frac{1}{E}\left(N_x\frac{\partial^2}{\partial x^2} + N_y\frac{\partial^2}{\partial y^2}\right)\right]w + \frac{1}{R^2}\frac{\partial^4 w}{\partial x^4} = 0 \qquad (16)$$

For uniform radial pressure, $N_x = N_{xy} = 0$, and

$$N_y = pR \qquad (17)$$

Thus

$$\left(\frac{1}{t_f}\frac{\partial^4}{\partial x^4} + \frac{E}{G\bar{t}}\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{1}{t_s}\frac{\partial^4}{\partial y^4}\right)\left(I_s\frac{\partial^4}{\partial x^4} + \right.$$

$$\left.\frac{GJ}{E}\frac{\partial^4}{\partial x^2 \partial y^2} + I_f\frac{\partial^4}{\partial y^4} + \frac{pR}{E}\frac{\partial^2}{\partial y^2}\right)(w) + \frac{1}{R^2}\frac{\partial^4 w}{\partial x^4} = 0 \qquad (18)$$

Let

$$w = w_{mn} \sin mx \sin ny \qquad (19)$$

Then equation (18) becomes

$$\left(\frac{m^4}{t_f} + \frac{Em^2n^2}{G\bar{t}} + \frac{n^4}{t_s}\right)\left(I_s m^4 + \frac{GJ}{E}m^2n^2 + I_f n^4 - p\frac{Rn^2}{E}\right) + \frac{m^4}{R^2} = 0 \qquad (20)$$

and consequently the pressure is found in the form

$$\frac{pRn^2}{E} = I_s m^4 + \frac{GJ}{E} m^2 n^2 + I_f n^4 + \frac{m^4}{R^2\left(\frac{m^4}{t_f} + \frac{Em^2n^2}{G\bar{t}} + \frac{n^4}{t_s}\right)} \tag{21}$$

or, with $\beta = n/m$, the expression for the pressure becomes

$$\frac{pR}{E} = m^2\left(\frac{I_s}{\beta^2} + \frac{GJ}{E} + I_f \beta^2\right) + \left[\left(R^2 m^2\right)\left(\frac{\beta^2}{t_f} + \frac{E}{G\bar{t}} \beta^4 + \frac{\beta^6}{t_s}\right)\right]^{-1} \tag{22}$$

Moderate-Length Cylinder Under Radial Pressure

The critical pressure for a moderate-length cylinder under radial pressure is obtained by minimization of pR, since R is constant. From $\frac{\partial(pR)}{\partial m} = 0$,

$$m = \left(\frac{I_s}{\beta^2} + \frac{GJ}{E} + I_f \beta^2\right)^{-1/4} \left(R^2\right)\left(\frac{\beta^2}{t_f} + \frac{E}{G\bar{t}} \beta^4 + \frac{\beta^6}{t_s}\right)^{-1/4}$$

or

$$\frac{1}{m} = (R)^{1/2}\left(I_s + \frac{GJ}{E} \beta^2 + I_f \beta^4\right)\left(\frac{1}{t_f} + \frac{E\beta^2}{G\bar{t}} + \frac{\beta^4}{t_s}\right)^{1/4} \tag{23}$$

which is the same as for an axially compressed orthotropic circular cylinder. When equation (23) is substituted into equation (22), it is found that

$$p = \frac{2E}{(R\beta)^2}\left(\frac{I_s + \frac{GJ}{E} \beta^2 + I_f \beta^4}{\frac{1}{t_f} + \frac{E}{G\bar{t}} \beta^2 + \frac{\beta^4}{t_s}}\right)^{1/2} \tag{24}$$

This is equal to σ_c/β^2, and becomes σ_c when $\beta = 1$.

Actually $m = \frac{a\pi}{L}$ and $n = \pi/\lambda$ where λ is the half-wave length in the circumferential direction. Then equation (22) becomes

$$\frac{pR}{E} = \left(\frac{a\pi}{L}\right)^2\left(\frac{I_s}{\beta^2} + \frac{GJ}{E} + I_f\beta^2\right) + \left[R^2\left(\frac{a\pi}{L}\right)^2\left(\frac{\beta^2}{t_f} + \frac{E}{G\bar{t}}\beta^2 + \frac{\beta^6}{t_s}\right)\right]^{-1} \tag{25}$$

For the minimum integral value of a in the solution of equation (25), $a = 1$. Then with $\beta \gg 1$ (where $\beta = L/\lambda$), which assumes that the axial wave length is much larger than the circumferential wave length,

$$\frac{pR}{E} = \left(\frac{\pi}{L}\right)^2 I_f\beta^2 + \left[R^2\left(\frac{\pi}{L}\right)^2 \frac{\beta^6}{t_s}\right]^{-1} \tag{26}$$

This is obtained by neglecting terms of lowest order in β. Then from $\frac{\partial(pR)}{\partial\beta} = 0$,

$$\beta^2 = \frac{L}{\pi}\left(\frac{3t_s}{R^2 I_f}\right)^{1/4} \tag{27}$$

and thus

$$\frac{pR}{E} = \frac{\pi}{L}\left(3^{1/4} + 3^{-3/4}\right)\left(\frac{t_s I_f{}^3}{R^2}\right)^{1/4} \tag{28}$$

Since $\sigma_y = \frac{pR}{t_f}$, then

$$\sigma_y = 5.51\,\frac{E}{L}\left(\frac{t_s I_f{}^3}{t_f{}^4 R^2}\right)^{1/4} \tag{29}$$

or

$$\sigma_y = 5.51 \frac{E}{L}\left[\left(\frac{t_s}{t_f}\right)\left(\frac{I_f}{t_f}\right)^3\left(\frac{1}{R^2}\right)\right]^{1/4} = 5.51E\left(\frac{t_s}{t_f}\right)^{1/4}\left(\frac{\rho_f}{R}\right)^{3/2}\left(\frac{R}{L}\right) \qquad (30)$$

For isotropic cylinders,

$$\sigma_y = 5.51 \frac{E}{L}\left(\frac{1}{12}\right)^{3/4}\left(\frac{t^3}{R}\right)^{1/2}$$

or

$$\sigma_y = 0.85E\left(\frac{t}{R}\right)^{3/2}\left(\frac{R}{L}\right) \qquad (31)$$

Using the notation

$$\sigma_y = \frac{k_y \pi^2 E t^2}{12\left(1 - \nu^2\right)L^2}$$

Then with $Z_L = \frac{L^2}{Rt}$, equation (31) reduces to

$$k_y = 1.033 {Z_L}^{1/2} \qquad (32)$$

(note that $\nu = 0$) which checks figure 1 of Batdorf's report (ref. 11), in which

$$k_y = 1.039 {Z_L}^{1/2} \qquad (33)$$

for $Z_L > 100$. This result can be obtained analytically. From Batdorf's report, for radial pressure on an isotropic cylinder,

$$k_y = \frac{\left(\bar{m}^2 + \beta^2\right)^2}{\beta^2} + \frac{12Z_L{}^2\bar{m}^4}{\pi^4\beta^2\left(\bar{m}^2 + \beta^2\right)^2} \tag{34}$$

where $m = \bar{m}\pi/L$. For hydrostatic pressure,

$$k_p = \frac{\left(\bar{m}^2 + \beta^2\right)^2}{\frac{\bar{m}^2}{2} + \beta^2} + \frac{12Z_L{}^2\bar{m}^4}{\pi^4\left(\bar{m}^2 + \beta^2\right)\left(\frac{\bar{m}^2}{2} + \beta^2\right)} \tag{35}$$

When $\bar{m} = 1$ and $\beta^2 \gg 1$,

$$k_y = k_p = \beta^2 + \frac{12Z_L{}^2}{\pi^4\beta^6} \tag{36}$$

Minimization of equation (36) leads to equation (33).

Moderate-Length Cylinder Under Hydrostatic Pressure

For the moderate-length orthotropic cylinder under hydrostatic pressure, equation (25) becomes

$$\frac{pR}{E}\left(\frac{m^2}{2} + n^2\right) = I_s m^4 + \frac{GJ}{E} m^2 n^2 + I_f n^4 + \frac{m^4}{R^2\left(\frac{m^4}{t_f} + \frac{Em^2n^2}{G\bar{t}} + \frac{n^4}{t_s}\right)} \tag{37}$$

or

$$\frac{pR}{E}\left(\beta^2 + \frac{1}{2}\right) = m^2\left(I_s + \frac{GJ}{E}\beta^2 + I_f\beta^4\right) + \left[R^2m^2\left(\frac{1}{t_f} + \frac{E}{G\bar{t}}\beta^2 + \frac{\beta^4}{t_s}\right)\right]^{-1} \tag{38}$$

When $\beta^2 \gg 1$, the results of hydrostatic and radial pressure are equal, and equation (30) will apply.

Long Cylinder Under Radial Pressure

For a long cylinder under radial pressure, the buckling mode corresponds to that of a ring, for which Levy obtained the result

$$p = 3EI_f/R^3 \tag{39}$$

With

$$\sigma_y = \frac{pR}{t_f} \tag{40}$$

it follows, from equations (39) and (40), that

$$\sigma_y = 3E\left(\rho_f/R\right)^2 \tag{41}$$

APPENDIX D

BUCKLING OF CIRCULAR CYLINDERS UNDER TORSION

Moderate-Length Cylinders

For shear loading of moderate-length circular cylinders

$$\left.\begin{aligned} N_{xy} &= t\tau = T/2\pi R^2 \\ N_x &= N_y = 0 \end{aligned}\right\} \tag{42}$$

As was shown by Becker and Gerard for isotropic cylinders of moderate length, a useful solution for τ_{cr} is obtained when the expression

$$w = \sin\left(\frac{m\pi x}{L} + \frac{ny}{R}\right) - \sin\left[(m + 2)\frac{\pi x}{L} + \frac{ny}{R}\right] \tag{43}$$

is used in equation (9). This satisfies $w = 0$ at the cylinder ends, although $\partial w/\partial x$ and $\partial^2 w/\partial x^2$ are not prescribed. Such freedom from boundary conditions is indicated for isotropic cylinders by the small difference (less than 10 percent) between the buckling stresses for clamped and simply supported cylinders of moderate length.

Equation (43) is adapted here for orthotropic cylinders. When it is substituted into equation (9), with the additional stipulations that $(m\pi R/nL)^2 \ll 1$ and $\left[(m + 2)\pi R/nL\right]^2 \ll 1$, then it is found that

$$\frac{2\tau}{E} = \left[\frac{LI_f}{(m + 1)\pi}\right]\left(\frac{n}{R}\right)^3 + \frac{t_s}{n^2}\left(\frac{\pi R}{nL}\right)^3\left[\frac{m^4 + (m + 2)^4}{2(m + 1)}\right] \tag{44}$$

The minimum value of τ is found from

$$\frac{\partial}{\partial m}\left(\frac{2\tau}{E}\right) = \frac{\partial}{\partial n}\left(\frac{2\tau}{E}\right) = 0$$

which yields

$$\frac{\tau}{E} = 3.46\left(\frac{t_s}{t}\right)^{3/8}\left(\frac{I_f}{R^2 t}\right)^{5/8}\left(\frac{R}{L}\right)^{1/2} \tag{45}$$

This agrees with the data of Stein, Sanders, and Crate for ring-stiffened cylinders in torsion (ref. 8), at large values of Z_L. It reduces to the isotropic solution for $\nu = 0$:

$$\frac{\tau}{E} = 0.731(t/R)^{5/4}(R/L)^{1/2} \tag{46}$$

The result of equation (46) is 5.7 percent lower than the exact isotropic solution obtained by Batdorf for $\nu = 0.313$.

Long Cylinders

For torsion on a long cylinder the classical solution for the isotropic case is

$$w = w_{mn} \sin\left(\frac{m\pi x}{L} + \frac{ny}{R}\right) \tag{47}$$

When this is substituted into equation (9) using equations (42), it is found that

$$\left(\frac{n}{R}\right)^8\left[\left(\frac{m\pi R}{nL}\right)^4 \frac{1}{t_f} + \left(\frac{m\pi R}{nL}\right)^2 \frac{E}{G\bar{t}} + \frac{1}{t_s}\right]\left[\left(\frac{m\pi R}{nL}\right)^4 I_s + \left(\frac{m\pi R}{nL}\right)^2 \frac{GJ}{E} + I_f\right] + \left(\frac{m\pi}{L}\right)^4 \frac{1}{R^2} =$$

$$\left(\frac{m\pi R}{nL}\right)\left(\frac{n}{R}\right)^6\left[\left(\frac{m\pi R}{nL}\right)^4 \frac{1}{t_f} + \left(\frac{m\pi R}{nL}\right)^2 \frac{E}{G\bar{t}} + \frac{1}{t_s}\right]2\tau t \tag{48}$$

Since $\nu = 0$, then $E/G = 2$. Furthermore, it is assumed again that $(m\pi R/nL) \ll 1$. Then equation (47) becomes

$$\frac{2\tau t}{E} = \left(\frac{nLI_f}{m\pi R}\right)\left(\frac{n}{R}\right)^2 + \left(\frac{m\pi R}{nL}\right)^3\left(\frac{t_s}{n^2}\right) \tag{49}$$

Upon minimization of τ with respect to m and employing $n = 2$ for the long cylinder,

$$\tau = 1.754E\left(\frac{I_f}{tR^2}\right)^{3/4}\left(\frac{t_s}{t}\right)^{1/4} \tag{50}$$

This reduces to the isotropic result, with $\nu = 0$,

$$\tau = 0.272E(t/R)^{3/2} \tag{51}$$

REFERENCES

1. Flügge, W.: Die Stabilität der Kreiszylinderschale. Ing.-Archiv, Bd. III, Heft 5, Dec. 1932, pp. 463-506.

2. Donnell, L. H.: Stability of Thin-Walled Tubes Under Torsion. NACA Rep. 479, 1933.

3. Dschou, Dji-Djüän: Die Druckfestigkeit versteifter zylindrischer Schalen. Luftfahrtforschung, Bd. 11, Nr. 8, Feb. 6, 1935, pp. 223-234.

4. Taylor, J. L.: Stability of a Monocoque in Compression. R. & M. No. 1679, British A.R.C., 1935.

5. Becker, Herbert: Handbook of Structural Stability. Part VI - Strength of Stiffened Curved Plates and Shells. NACA TN 3786, 1958.

6. Salerno, V. L., and Levine, B.: General Instability of Reinforced Shells under Hydrostatic Pressure. Polytechnic Inst. of Brooklyn, Rep. No. 189, Sept. 1951.

7. Kendrick, S.: The Buckling Under External Pressure of Circular Cylindrical Shells with Evenly Spaced, Equal Strength, Circular Ring Frames. Parts I, II, and III. Reps. 211, 243, and 244, Naval Construction Res. Establishment (Gt. Britain), 1955.

8. Bodner, S. R.: The Analysis of the General Instability of Ring Reinforced Circular Cylindrical Shells by Orthotropic Shell Theory. Rep. 291, Contract N6-onr-26303, Office of Naval Res. and Polytechnic Inst. of Brooklyn, May 1955.

9. Stein, Manuel, Sanders, J. Lyell, Jr., and Crate, Harold: Critical Stress of Ring-Stiffened Cylinders in Torsion. NACA Rep. 989, 1950.

10. Hayashi, T.: Torsional Buckling of Orthogonal Anisotropic Cylinders. 8th Int. Congress of Applied Mechanics, Aug. 1952 (Reprint).

11. Batdorf, S. B.: A Simplified Method of Elastic-Stability Analysis for Thin Cylindrical Shells. I-Donnell's Equation. NACA TN 1341, 1947.

12. Levy, M.: Liouville's Jour. Pure and Appl. Math., Ser. 3, vol. 10, 1884, p. 5 ff.

13. Becker, H., and Gerard, G.: Torsional Buckling of Moderate Length Cylinders. Jour. Appl. Mech., vol. 23, no. 4, Dec. 1956, pp. 647-648.

TABLE 1

THEORETICAL GENERAL INSTABILITY STRESSES FOR

ORTHOTROPIC CIRCULAR CYLINDERS

Loading	Moderate-length cylinders	Long cylinders
Axial compression	[a] $\sigma_c = \frac{2E}{Rt_s}\left[\frac{\beta^2 I_f + (GJ/2E)}{\left(\beta^2/t_s\right) + \left(E/2G\bar{t}\right)}\right]^{1/2}$	[a] $\sigma_c = \frac{2E}{Rt_s}\left[\frac{\beta^2 I_f + (GJ/2E)}{\left(\beta^2/t_s\right) + \left(E/2G\bar{t}\right)}\right]^{1/2}$
External radial or hydrostatic pressure	$\sigma_y = 5.51E\left(\frac{t_s}{t_f}\right)^{1/4}\left(\frac{\rho_f}{R}\right)^{3/2}\left(\frac{R}{L}\right)$	$\sigma_y = 3E\left(\rho_f/R\right)^2$
Torsion	$\tau = 3.46E\left(\frac{t_s}{t}\right)^{3/8}\left(\frac{I_f}{R^2 t}\right)^{5/8}\left(\frac{R}{L}\right)^{1/2}$	$\tau = 1.754E\left(\frac{t_s}{t}\right)^{1/4}\left(\frac{I_f}{R^2 t}\right)^{3/4}$

[a] See equation (14) for value of β^2.

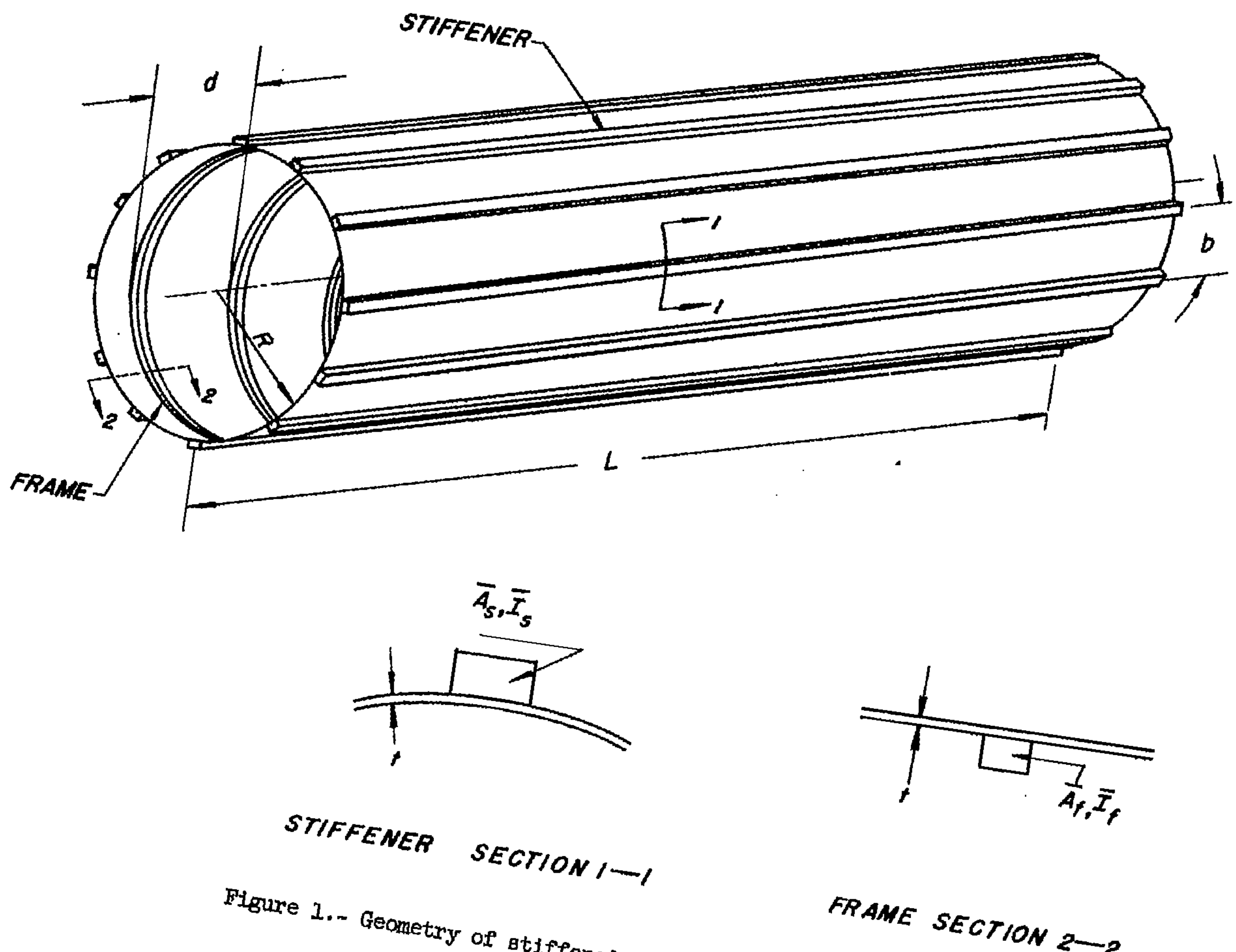

Figure 1.- Geometry of stiffened circular cylinder.

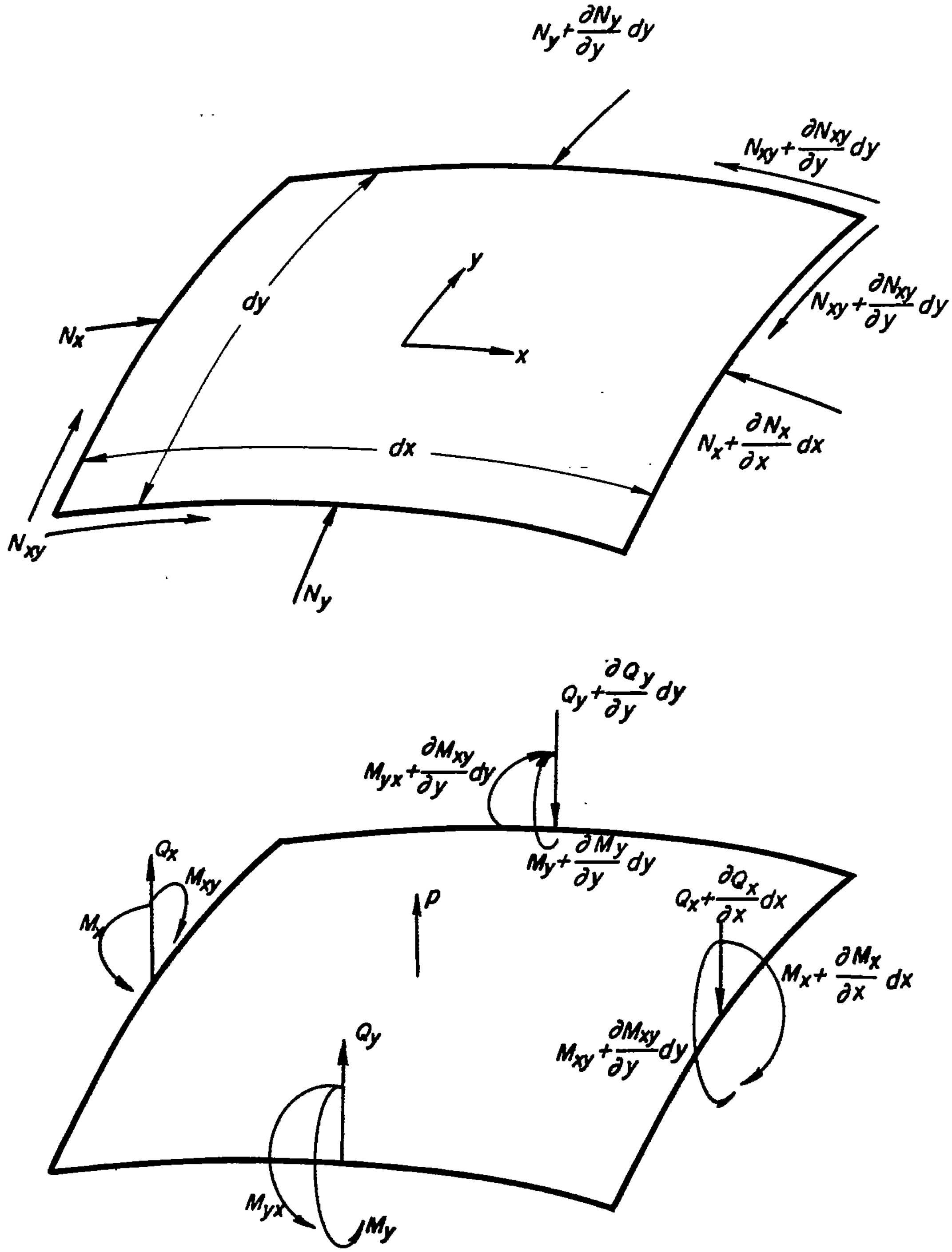

Figure 2.- Forces acting on a cylinder element.